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Quasi-Exactly-Solvable Many-Body Problems

by

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ABSTRACT

Explicit examples of quasi-exactly-solvable N -body problems on the line are presented. These are related to the hidden algebra sl_N , and they are of two types – containing up to N (infinitely-many eigenstates are known, but not all) and up to 6 body interactions only (a finite number of eigenstates is known). Both types degenerate to the Calogero model.

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The Calogero model [1] is one of the most remarkable objects in non-relativistic multidimensional quantum mechanics. Moreover, a quite exciting relation of this model with to the two-dimensional Yang-Mills theory has been found recently [2]. The Calogero model has many beautiful properties such as: complete-integrability, maximal super-integrability and being an exactly-solvable N -body problem on the real line. The model is defined by the Hamiltonian

$$H_{Cal} = \frac{1}{2} \sum_{i=1}^N \left[-\partial_i^2 + \omega^2 x_i^2 \right] + \sum_{j < i}^N \frac{g}{(x_i - x_j)^2} + V^* , \quad (1)$$

where $\partial_i \equiv \frac{\partial}{\partial x_i}$, $V^* = 0$, ω is the harmonic oscillator frequency, hereafter normalized to $\omega = 1$, and $g > -1/8$ is the coupling constant. Recently, it was found [3] that the Calogero model is characterized by the hidden algebra sl_N and that the Calogero Hamiltonian (1) is a Lie-algebraic, exactly-solvable operator. The goal of this Letter is to show that the Calogero Hamiltonian can be generalized to a Lie-algebraic, *quasi-exactly-solvable* operator leading to explicit examples of quasi-exactly-solvable N -body problems.

For the above purpose, let us first make a gauge rotation in equation (1) taking the Calogero ground-state wave function as a gauge factor [4]. We have:

$$\begin{aligned} h &= -2 \Psi_0^{-1} H_{Cal} \Psi_0 \equiv -2\beta(x)^{-\nu} e^{\frac{1}{2} \sum^N x_i^2} H_{Cal} \beta(x)^\nu e^{-\frac{1}{2} \sum^N x_i^2} = \\ &= \sum_{i=1}^N \partial_i^2 - 2 \sum_{i=1}^N x_i \partial_i + \nu \sum_{j \neq i}^N \frac{1}{x_i - x_j} [\partial_i - \partial_j] \\ &\quad - N - \nu N(N-1) + 2V^* , \end{aligned} \quad (2)$$

where ν is one of two solutions to the equation $g = \nu(\nu - 1)$, and $\beta(x) = \prod_{i>j} (x_i - x_j)$ is the Vandermonde determinant. For the sake of simplicity, from hereon we shall omit the constant term in (2), since it only shifts the reference point of the spectrum. As the next step, we introduce the translation-invariant elementary symmetric polynomials [3] :

$$\tau_n(x) = \sigma_n(y(x)) , \quad n = 2, 3, \dots, N , \quad (3)$$

where $\sigma_n(z)$ are the standard elementary symmetric polynomials (see, for example, [5]),

$$Y = \sum_{j=1}^N x_j, \quad y_i = x_i - \frac{1}{N} \sum_{j=1}^N x_j, \quad i = 1, 2, \dots, N, \quad (4)$$

with the constraint $\sum_{i=1}^N y_i = 0$, and Y is the center-of-mass coordinate. Making the change of variables¹

$$(x_1, x_2, \dots, x_N) \rightarrow (Y, \tau_n(x) | n = 2, 3, \dots, N),$$

the operator h , after extraction of the center of mass motion, transforms into (see [3]):

$$\begin{aligned} h_{rel} = & \sum_{j,k=2}^N A_{jk} \frac{\partial^2}{\partial \tau_j \partial \tau_k} - 2 \sum_{j=2}^N j \tau_j \frac{\partial}{\partial \tau_j} - \left(\frac{1}{N} + \nu \right) \sum_{j=2}^N (N-j+2)(N-j+1) \tau_{j-2} \frac{\partial}{\partial \tau_j} \\ & + V^*(\tau), \end{aligned} \quad (5)$$

where

$$A_{jk} = \frac{(N-j+1)(k-1)}{N} \tau_{j-1} \tau_{k-1} + \sum_{\ell \geq \max(1, k-j)} (k-j-2\ell) \tau_{j+\ell-1} \tau_{k-\ell-1}$$

Here we put $\tau_0 = 1, \tau_1 = 0$ and $\tau_p = 0$, if $p < 0$ and $p > N$. It is worth noting that the Calogero Hamiltonian (1) is Z_2 -invariant: $(x_i \rightarrow -x_i)$; the τ -variables are (anti)symmetric under this transformation: $\tau_n(-x_1, -x_2, \dots, -x_N) = (-)^n \tau_n(x_1, x_2, \dots, x_N)$.

Now let us introduce the real algebra $sl_N(\tau)$ of first-order differential operators in the most degenerate representation, where all spins vanish except one :

$$J_i^- = \frac{\partial}{\partial \tau_i}, \quad i = 2, 3, \dots, N,$$

$$J_{i,j}^0 = \tau_i J_j^- = \tau_i \frac{\partial}{\partial \tau_j}, \quad i, j = 2, 3, \dots, N,$$

¹For this transformation the Jacobian in explicit form is not known so far

$$J^0(n) = n - \sum_{p=2}^N \tau_p \frac{\partial}{\partial \tau_p} , \quad (6)$$

$$J_i^+(n) = \tau_i J^0 , \quad i = 2, 3, \dots, N .$$

If the parameter n is a non-negative integer number, the representation (6) becomes finite-dimensional and its representation space is given by the space of polynomials

$$\mathcal{P}_n = \text{span}\{\tau_2^{n_2} \tau_3^{n_3} \tau_4^{n_4} \dots \tau_N^{n_N} : 0 \leq n_i \leq n\} . \quad (7)$$

It is worth noting that $J_{i,j}^0$ form the algebra $sl_{N-1} \subset sl_N$ and, in turn, J_i^- , $J_{i,i}^0$ form the sub-algebra $b_2 \subset sl_N$.

It is evident that when $V^*(\tau) = 0$ the operator h_{rel} can be rewritten in terms of the generators of $sl_N(\tau)$ given by (6) and, furthermore, it does not contain the generators J_i^+ . This implies that the operator h_{rel} is exactly-solvable, i.e. this operator preserves the flag of spaces $\mathcal{P}_n : \mathcal{P}_0 \subset \mathcal{P}_1 \subset \mathcal{P}_2 \dots$ (see [6] and also [3]).

We now proceed to study the eigenfunctions of the operator h_{rel} given in (5). Because $A_{22} = -2\tau_2$ depends on τ_2 only, then, for the case $V^* = V^*(\tau_2)$, we have the remarkable property of the existence of a family of eigenfunctions of h_{rel} depending on τ_2 only. Due to this property the original eigenvalue problem

$$h_{rel}\phi(\tau_2) = \epsilon\phi(\tau_2) , \quad (8)$$

is simplified to

$$h_{rel}(\tau_2)\phi(\tau_2) = \epsilon\phi(\tau_2) , \quad (9)$$

where

$$h_{rel}(\tau_2) = -2\tau_2 \frac{\partial^2}{\partial \tau_2^2} - (4\tau_2 + b_2) \frac{\partial}{\partial \tau_2} + 2V^*(\tau_2) . \quad (10)$$

Or, in terms of the generators (6),

$$h_{rel}(\tau_2) = -2J_{2,2}^0 J_2^- - 4J_{2,2}^0 - b_2 J_2^- + 2V^*(\tau_2) \quad (11)$$

with

$$b_2 = (1 + \nu N)(N - 1) .$$

For the case of the Calogero model, $V^* = 0$, it is easy to find the eigenfunctions and eigenvalues of (9)–(10) in the form:

$$\phi^{\{k\}}(\tau_2) = L_k^{(\frac{b_2}{2}-1)}(-2\tau_2), \quad \epsilon_k = -4k, \quad k = 0, 1, 2, \dots, \quad (12)$$

where $L_k^{(\alpha)}$ are the associated Laguerre polynomials.

In order to carry out a Lie-algebraic analysis of $h_{rel}(\tau_2)$ in equation (10), note that the algebra $sl_N(\tau)$ contains the sub-algebra $b_2(\tau_2)$ formed by $J_{2,2}^0, J_2^-$: $b_2(\tau_2) \subset sl_N(\tau)$. This sub-algebra $b_2(\tau_2)$ can be extended to $sl_2(\tau_2)$ with generators given by

$$J^+(n) = \tau_2^2 \frac{\partial}{\partial \tau_2} - n\tau_2, \quad J^0(n) = \tau_2 \frac{\partial}{\partial \tau_2} - \frac{n}{2}, \quad J^- = \frac{\partial}{\partial \tau_2}, \quad (13)$$

such that for $n = 0$: $J^0(0) = J_{2,2}^0, J_2^- = J^-$. It is worth emphasizing that in this realization $sl_2(\tau_2) \not\subset sl_N(\tau)$. So in terms of the generators (13), the operator (10) takes the form

$$h_{rel}(\tau_2) = -2J^0(n)J^- - 4J^0(n) - (b_2 + n)J^- - 2n + 2V^*(\tau_2); \quad (14)$$

(cf.(11)).

One can now pose the following natural question: could we gauge-rotate $h_{rel}(\tau_2)$ in the τ_2 direction and fit $V^*(\tau_2)$ in such a way as to obtain a Lie algebraic operator. In concrete terms this means that we want to find $\Psi^*(\tau_2)$ and $V^*(\tau_2)$ such that

$$(\Psi^*(\tau_2))^{-1} h_{rel}(\tau_2) \Psi^*(\tau_2) \in U_{sl_2(\tau_2)}, \quad (15)$$

where $U_{sl_2(\tau_2)}$ denotes the universal enveloping algebra sl_2 taken in the representation (13). We have been able to find three concrete examples which provide an affirmative answer to this question and lead to three types of *quasi-exactly-solvable, many-body* problems.

(I). Take as a gauge factor $\Psi^*(\tau_2) = \tau_2^\alpha$ and choose $V^* = \frac{\alpha}{\tau_2}$. Then it is easy to see that the gauge-rotated operator h_{rel} remains Lie-algebraic:

$$\begin{aligned} h_{rel}^{(1)}(\tau_2) &= \tau_2^{-\alpha} h_{rel}(\tau_2) \tau_2^\alpha \\ &= -2J^0(n)J_{(n)}^- - 4J^0(n) - (b_2 + n + 4\alpha)J_{(n)}^-, \end{aligned} \quad (16)$$

provided that

$$2\gamma = \alpha(b_2 + 2\alpha - 2) . \quad (17)$$

The resulting modified Calogero Hamiltonian, which in this case contains up to N body interactions, is given by:

$$H_{Cal}^{(1)} = \frac{1}{2} \sum_{i=1}^N \left[-\partial_i^2 + x_i^2 \right] + \sum_{j < i}^N \frac{g}{(x_i - x_j)^2} + \frac{\gamma}{\tau_2} , \quad (18)$$

with the eigenfunctions

$$\Psi = \beta(x)^\nu e^{-\frac{\gamma^2}{2}\tau_2} \tau_2^\alpha \begin{cases} L_k^{(\frac{b_2}{2}+2\alpha-1)}(-2\tau_2) \\ \phi_{\{k\}}(\tau_2, \tau_3, \dots) \end{cases} , \quad (19)$$

and the eigenvalues

$$\epsilon_k = -4k - 4\alpha , \quad (20)$$

(cf.(12)), corresponding to the eigenfunctions defined by Laguerre polynomials. As before $\beta(x)$ is the Vandermonde determinant. Unlike what occurs in the original Calogero model ($V^* = 0$), the remaining eigenfunctions $\phi_{\{k\}}(\tau_2, \tau_3, \dots)$ are not polynomials anymore. Note in passing that for the particular case $N = 2$, the Hamiltonian $H_{Cal}^{(1)}$ becomes the well-known Kratzer Hamiltonian (see, for example, [7], problem 69).

A simple analysis shows that the requirement of normalizability (square-integrability at the origin) of the τ_2 -family of eigenfunctions (19) leads to the constraint

$$\alpha > -\frac{b_2}{4} . \quad (21)$$

Thus, this deformation of the Calogero model allows us to find infinitely-many eigenstates explicitly but not all of them. This situation is reminiscent of that occurring in the case of the Hulthen and Saxon-Woods potentials, where the s -states can be found explicitly but not all other states (see, for example, Flugge [7], problems 64, 68).

(II). Another case leading to a truly quasi-exactly-solvable modification of the Calogero model can be constructed by generalizing the previous example. Take as a gauge factor

$$\Psi^{**}(\tau_2) = \tau_2^\alpha \exp\left(-\frac{a}{2}\tau_2^2 - b\tau_2\right)$$

and choose $V^{**} = \frac{\gamma}{\tau_2} + A\tau_2^3 + B\tau_2^2 + C\tau_2$ with appropriate coefficients. Then the gauge-rotated operator $h_{rel}(\tau_2)$ remains Lie-algebraic:

$$\begin{aligned} h_{rel}^{(2)}(\tau_2) &= \tau_2^{-\alpha} \exp\left(\frac{a}{2}\tau_2^2 + b\tau_2\right) h_{rel}(\tau_2) \tau_2^\alpha \exp\left(-\frac{a}{2}\tau_2^2 - b\tau_2\right) \\ &= -2\tau_2 \frac{\partial^2}{\partial \tau_2^2} + \left[4a\tau_2^2 + 4(b-1)\tau_2 - b_2\right] \frac{\partial}{\partial \tau_2} - 4an\tau_2 - 2n(b-1) \\ &= -2J^0(n)J^- + 4aJ^+(n) + 4(b-1)J^0(n) - (b_2 + n + 4\alpha) J^- , \end{aligned} \quad (22)$$

(cf.(16)). The corresponding modified Calogero Hamiltonian is of the form

$$H_{Cal}^{(2)} = \frac{1}{2} \sum_{i=1}^N \left[-\partial_i^2 + x_i^2 \right] + \sum_{j<i}^N \frac{g}{(x_i - x_j)^2} + \frac{\gamma}{\tau_2} + A\tau_2^3 + B\tau_2^2 + C\tau_2, \quad (23)$$

provided that

$$\begin{aligned} A &= a^2 , \\ B &= 2a(b-1) , \\ C &= \left[(b-1)^2 - 1 - a \left(2n + 1 + 2\alpha + \frac{b_2}{2} \right) \right] , \end{aligned}$$

with γ given by (17). The eigenfunctions of this new Hamiltonian are:

$$\Psi = \beta(x)^\nu e^{-\frac{\gamma^2}{2}} \tau_2^\alpha \exp\left(-\frac{a}{2}\tau_2^2 - b\tau_2\right) \begin{cases} p_n^{\{k\}}(\tau_2) , k = 0, 1, 2, \dots n \\ \phi_{\{k\}}(\tau_2, \tau_3, \dots) \end{cases} , \quad (24)$$

where the $p_n^{\{k\}}$ are polynomials of degree n , while generically the $\phi_{\{k\}}$ are not polynomials. In order to ensure the normalizability of (24), the parameter α should obey the constraint (21). Note that for the two-body case, the Hamiltonian $H_{Cal}^{(2)}$ corresponds to one of the well-known examples of one-dimensional quasi-exactly-solvable problems [8]. For the general N -body case if $\alpha = 0, b = 1$ the Hamiltonian (23) coincides with that obtained in [9]. Let us emphasize that when $\alpha = 0$ and, correspondingly $\gamma = 0$, the quasi-exactly-solvable Hamiltonian (23) contains two, three, and up to six-body interactions only, independently on the number of bodies N . This follows immediately from the identity:

$$2N\tau_2 = - \sum_{i>j} (x_i - x_j)^2 .$$

The problem of finding the polynomials $p_n^{\{k\}}(\tau_2)$ in (24) is reduced to solving an algebraic equation of degree n whose roots are the corresponding eigenvalues $\epsilon_n^{\{k\}}$. We just present the explicit formulae for $n = 0, 1, 2$. For the sake of simplicity, we take $b = 1$.

It is clear that for $n = 0$ the polynomial eigenfunction is a constant, $p_0^{\{0\}} = \text{const}$ and $\epsilon_0^{\{0\}} = 0$. For $n = 1$ the two polynomial eigenfunctions have the form

$$p_1^{\pm} = \tau_2 - \frac{\epsilon_1^{\pm}}{4a},$$

where $\epsilon_1^{\pm} = \pm 2\sqrt{ab_2}$ are the corresponding eigenvalues. Note that $p_1^{\pm}(\epsilon_1^{\pm})$ form a double-sheeted Riemann surface in the parameter space $a(b_2)$ in complete agreement with [8]. For the case $n = 2$ the eigenfunctions are quadratic polynomials in τ_2 and are given by

$$\begin{aligned} p_2^{\{1\}} &= \tau_2^2 - \frac{2+b_2}{4a}, \\ p_2^{\{0,2\}} &= \tau_2^2 + \epsilon_2^{\{0,2\}}\tau_2 + \frac{b_2}{4a}, \end{aligned} \quad (25)$$

with the corresponding eigenvalues

$$\begin{aligned} \epsilon_2^{\{1\}} &= 0, \\ \epsilon_2^{\{0,2\}} &= \pm 4\sqrt{a(1+b_2)}. \end{aligned} \quad (26)$$

Observe that the number of zeroes of $p_2^{\{k\}}$, which is equal to k , agrees with the Sturm theorem.

(III). In order to proceed to another example of quasi-exactly-solvable many-body problems let us mention that since $A_{23} = -3\tau_3$ the original Calogero model (8) has, besides the eigenfunctions (12), another outstanding family of eigenfunctions of the form

$$\phi^{\{k\}}(\tau_2, \tau_3) = \tau_3 L_k^{(\frac{b_2}{2}+2)}(-2\tau_2), \quad (27)$$

with eigenvalues

$$\epsilon_k = -4k - 6, \quad k = 0, 1, 2, \dots$$

This functional form suggests to take as a gauge factor

$$\Psi^{***}(\tau_2, \tau_3) = \tau_3 \tau_2^\alpha \exp\left(-\frac{a}{2}\tau_2^2 - b\tau_2\right),$$

and the same functional form V^{**} for the potential as in the previous section. Now we make the gauge transformation of the original h_{rel} in (5) with this factor. Remarkably, we find that the resulting operator (after a suitable choice of the parameters in V^{**}) still has eigenfunctions depending on τ_2 only ! Moreover, it is easy to see that the τ_2 -depending operator $h_{rel}^{(3)}(\tau_2)$, defining these τ_2 -depending eigenfunctions, is given by

$$\begin{aligned} h_{rel}^{(3)}(\tau_2) &= \\ &= -2\tau_2 \frac{\partial^2}{\partial \tau_2^2} + [4a\tau_2^2 + 4(b-1)\tau_2 - (b_2 + 6)] \frac{\partial}{\partial \tau_2} - 4an\tau_2 - 2n(b-1) - 6. \end{aligned} \quad (28)$$

The Lie-algebraic form of (28) is

$$h_{rel}^{(3)}(\tau_2) = -2J^0(n)J^- + 4aJ^+(n) + 4(b-1)J^0(n) - [(b_2 + 6) + n + 4\alpha] J^- - 6, \quad (29)$$

(cf.(16), (22)). The corresponding modified Calogero Hamiltonian $H_{Cal}^{(3)}$ coincides with (23) with the following slight modifications in the parameters: b_2 is replaced by $b_2 + 6$, and the reference point of the spectrum is shifted by (-6). The corresponding eigenfunctions are then given by

$$\Psi = \beta(x)^\nu e^{-\frac{Y^2}{2}} \tau_2^\alpha \tau_3 \exp\left(-\frac{a}{2}\tau_2^2 - b\tau_2\right) \begin{cases} p_n^{\{k\}}(\tau_2), & k = 0, 1, 2, \dots, n \\ \phi_{\{k\}}(\tau_2, \tau_3, \dots) \end{cases}, \quad (30)$$

(cf.(24)), where the $p_n^{\{k\}}$ are polynomials of degree n , while generically the $\phi_{\{k\}}$ are again not polynomials. The normalizability of (30) dictates that the parameter α should obey the constraint (21) with $b_2 \rightarrow (b_2 + 6)$. Also the expressions for $p_n^{\{k\}}$ obtained in (25)–(26) remain valid replacing $b_2 \rightarrow (b_2 + 6)$. It is worth noticing that when $N = 2$, the whole family of eigenfunctions (30) vanishes.

In conclusion, we showed the three cases of quasi-exactly-solvable many-body problems on the line characterizing up to N -body interactions (I), up to 6-body interactions (II)–(III) and having no $N = 2$ limit (III). All of them are associated with the Calogero model. These examples are described by the Hamiltonian:

$$H_{Cal}^{QES} = \frac{1}{2} \sum_{i=1}^N \left[-\partial_i^2 + x_i^2 \right] + \sum_{j < i}^N \frac{g}{(x_i - x_j)^2} + \frac{\gamma}{\tau_2} + A\tau_2^3 + B\tau_2^2 + C\tau_2,$$

where the parameters are

$$\begin{aligned}
A &= a^2 & (31) \\
B &= 2a(b-1), \\
C &= \left[(b-1)^2 - 1 - a \left(2n + 1 + 2\alpha + \frac{b_2}{2} + 3\mu \right) \right] \\
2\gamma &= \alpha(b_2 + 6\mu + 2\alpha - 2) .
\end{aligned}$$

The parameter μ can be either 0 or 1 leading to solutions of the form (24) or (30), respectively. The parameter a is non-negative, while b can be any real number, and the parameter α is restricted to satisfy

$$\alpha > -\frac{b_2 + 6\mu}{4} . \quad (32)$$

If $a, b = 0$, we return to the Hamiltonian (18) with solutions (19).

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